A QUIVER REPRESENTATION WITH NONREALIZABLE POINTS AND TRIVIAL VALUATION IN DIMENSION 3

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We construct a quiver and quiver representation such that the quiver Grassmannian is strictly contained in the quiver Dressian for $n = 3$ over a field with trivial valuation.

We obtain a nonrealizable quiver representation using the nonrealizable Q-representation R for $n = 4$, from [1, Example 4.4], as follows. Inspired by the fact that a matroid over [4] can be written as the matroid quotient of two matroids over [3], we replace each vertex V_i of Q with a new quiver, as depicted in Figure 1, obtaining the quiver Q'.

Now, Q' is given as four copies of the quiver (a) in Figure 1, which we call *layers* $L_1, \ldots L_4$. Each vertex in L_i is connected with an arrow to the vertex in the same position in L_{i+1} (modulo 4).

We define the Q'-representation D as follows. We assign to each vertex of Q' the vector space $\mathbb{C}^3$, and to each arrow inside a layer (i.e. all the arrows in Figure 1(a)) the identity map. To each arrow connecting two layers L_i and L_{i+1} (i.e. the vertical arrows in Figure 1(b) and (c)), we assign a rank 2 projection such that the composition with the incoming map connecting L_{i-1} and L_i has rank 1. Denoting by $V_{i,j}$ the red vertices in 1(b) and by $W_{i,j}$ the blue ones, we assign the same linear map to the arrows $\alpha : V_{i,j} \to V_{i,j+1}$ and $\alpha : W_{i,j} \to W_{i,j+1}$ that correspond to moving up layers.

Finally, we consider a dimension vector $\mathbf{d}$ whose entries are in $\{1, 2\}$, corresponding to the vertices depicted in Figure 1(a)) – red and black vertices are assigned dimension 1, blue vertices are assigned dimension 2.

We now show that the quiver Grassmannian $\mathrm{QGr}(D, \mathbf{d}; 3)$ has the same Plücker relations as the quiver Grassmannian in [1, Example 4.4].

Each arrow in a layer corresponds to a flag of dimension $(1, 2)$ and thus produces exactly one three-term Plücker relation, of the form $p_{v,1}p_{w,23} - p_{v,2}p_{w,13} + p_{v,3}p_{w,12}$. For each red or black vertex (corresponding to a 1-dimensional subspace), the arrow connecting the layers leaves two Plücker variables p_i and p_j unchanged, and replaces the third variable p_k by a new variable p_l. For each blue vertex (corresponding to a 2-dimensional subspace), the arrow fixes the Plücker variable p_{ij} and replaces the variables p_{ik} and p_{jk} with p_{il} and p_{jl} respectively.

Moving up along the four layers, since the layers form a cycle, the only four variables arising from subspaces of dimension 1 are exactly p_1, p_2, p_3 and p_4. For the rightmost vertex in Figure 1 (a), as $G(1, 4) \cong G(3, 4)$, we can rename the four variables $p_{123}, p_{124}, p_{134}$ and p_{234}.

The six variables arising from subspaces of dimension 2 corresponding to the blue vertices are $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}$ and p_{34}. Since the layer maps for the vertices $V_{i,j}$ and $W_{i,j}$ corresponding to the red and blue vertices are equal, we can interpret the coordinates associated to the red vertices exactly as the complementary ones associated to the blue vertices on the same layer. For instance, consider the following subgraph of Figure 1(b), as depicted in (d).

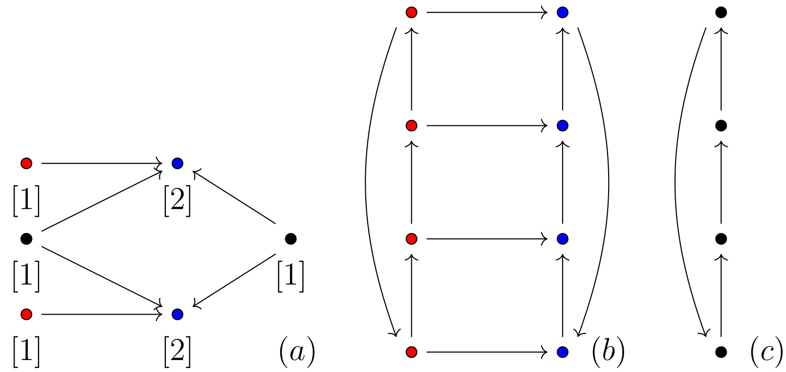

FIGURE 1. Replacement of vertices. (a) depicts one layer of the resulting quiver Q', (b) depicts the quiver replacing the middle vertices in [1, Example 4.4], i.e. the vertices which were assigned subspaces of dimension 2, and (c) depicts the replacement of the other vertices.

$$\begin{tikzcd} \{1,2,4\} \arrow[r, "A"] & \{1,2,4\} \\ \{1,2,3\} \arrow[u, "\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}"] \arrow[r, "B"'] & \{1,2,3\} \arrow[u, "\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}"'] \end{tikzcd} \qquad (d)$$

The identity map A induces the quiver Plücker relation $p_1p_{23} - p_2p_{13} + p_3p_{12}$. Now, we rename the Plücker variables p_i to p_{i4} for all $i \in \{1,2,3\}$ and obtain the relation $p_{14}p_{23} - p_{24}p_{13} + p_{34}p_{12}$. Analogously, for the identity map B we have the relation $p_1p_{24} - p_2p_{14} + p_4p_{12}$, and we replace the variables p_i by p_{i3} for all $i \in \{1,2,4\}$, yielding again the relation $p_{14}p_{23} - p_{24}p_{13} + p_{34}p_{12}$, i.e. both maps A and B induce the same Plücker relation.

Now, by the construction of Q', we obtain exactly the same Plücker relations as in [1, Example 4.4]. Thus, the two quiver representations with the respective dimension vectors produce the same tropicalized quiver Grassmannian and quiver Dressian, and hence $\overline{\mathrm{trop}}(\mathrm{QGr}(D, \mathbf{d}; 3) \neq \mathrm{QDr}(D, \mathbf{d}; 3)$.

REFERENCES

[1] Giulia Iezzi and Victoria Schleis. Tropical Quiver Grassmannians, 2023.